# Integration on Measure Chains

BERND AULBACH and LUDWIG NEIDHART
Department of Mathematics, University of Augsburg
D-86135 Augsburg, Germany
E-mail: aulbach@math.uni-augsburg.de
neidhart@math.uni-augsburg.de

**Abstract** In its original form the calculus on measure chains is mainly a differential calculus. The notion of integral being used, the so-called Cauchy integral, is defined by means of antiderivatives and, therefore, it is too narrow for the development of a full infinitesimal calculus. In this paper, we present several other notions of integral such as the Riemann, the Cauchy-Riemann, the Borel and the Lebesgue integral for functions from a measure chain to an arbitrary real or complex Banach space. As in ordinary calculus, of those notions only the Lebesgue integral provides a concept which ensures the extension of the original calculus on measure chains to a full infinitesimal calculus including powerful convergence results and complete function spaces.

**Keywords** Measure chain, Time scale, Cauchy integral, Riemann integral, Cauchy-Riemann integral, Borel integral, Lebesgue integral, Bochner integral

**AMS Subject Classification** 26A42, 28A25, 28B05, 39A10, 39A12

## 1 Introduction

The creation of *measure chains* was motivated by the long standing desire to have some means which allow to treat problems arising in the theory of differential and/or difference equations in a unified way. With this aim in mind, Stefan Hilger introduced in his PhD thesis [7] (see also [8]) the concept of a *measure chain* and developed a rather complete theory of differentiation for functions which are defined on a measure chain (or a subset of it) and have their values in an arbitrary real or complex Banach space. As far as integration of those functions is concerned, he mentioned the possibility of a measure theoretic approach, however, (in favor of an application of his theory to a prototype problem on invariant manifolds) he confined his study to a basic notion of integral which is simply defined by means of antiderivatives. The corresponding notion of integral, the so-called *Cauchy integral*, turned out to be sufficient for the achievement of the original goals, and even in the long run, it has proved to be a successful concept leading to many research activities providing interesting new results in the field of dynamic equations on measure chains (among many others, ten papers in this volume).

While most of these activities focus on *applications* of the calculus on measure chains to dynamic equations, the *foundations* of this calculus have obtained only minor attention. In some papers on dynamic equations, the calculus itself has been promoted, but only as much as it was needed for a certain purpose (for the *chain rule*, e.g., see Keller [10] and Pötzsche [13]). As to our knowledge, the only papers (or theses) dealing with the *theory of integration* on measure chains are Sailer [14], Neidhart [12], Guseinov and Kaymakçalan [9] and Bohner and Peterson [4, Chapter 5].

In this paper, we give a survey of the main results contained in the diploma thesis [12] of the second author. In fact, we outline the definitions of the Cauchy, the Riemann, the Cauchy-Riemann, the Borel and the Lebesgue integral for functions from a measure chain into an arbitrary real or complex Banach space; and we briefly sketch their mutual interrelations. For further information on this topic we refer the reader to Neidhart [12].

## 2 Basic fact about measure chains

For the reader's convenience we briefly state some facts on measure chains which are used in this paper. For more details we refer to Hilger [8] and Bohner & Peterson [3, Section 8.1].

A *measure chain* is a triple $(\mathbb{T}, \leq, \mu)$ consisting of a set $\mathbb{T}$, a relation $\leq$ on $\mathbb{T}$ and a function $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, the so-called *growth calibration*, such that the following axioms hold:

Axioms on $\leq$ :

| | | |
|---|---|---|
| (Reflexivity) | $\forall\, x \in \mathbb{T}:$ | $x \leq x$ |
| (Antisymmetry) | $\forall\, x, y \in \mathbb{T}:$ | $x \leq y \leq x \;\Rightarrow\; x = y$ |
| (Transitivity) | $\forall\, x, y, z \in \mathbb{T}:$ | $x \leq y \leq z \;\Rightarrow\; x \leq z$ |
| (Totality) | $\forall\, x, y, z \in \mathbb{T}:$ | $x \leq y$ or $y \leq x$ |
| (Completeness) | Any non-void subset of $\mathbb{T}$ which is bounded above has a least upper bound | |

Axioms on $\mu$ :

| | | |
|---|---|---|
| (Cocycle property) | $\forall\, x, y, z \in \mathbb{T}:$ | $\mu(x, y) + \mu(y, z) = \mu(x, z)$ |
| (Strong isotony) | $\forall\, x, y \in \mathbb{T}:$ | $x > y \;\Rightarrow\; \mu(x, y) > 0$ |
| (Continuity) | $\mu$ is continuous | |

In order to fix (possibly ambiguous) notation we denote by $\sigma : \mathbb{T} \to \mathbb{T}$ the *forward jump operator*, i.e. $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ on $\mathbb{T}$, and by $\rho : \mathbb{T} \to \mathbb{T}$ the *backward jump operator* i.e. $\rho(t) := \sup\{s \in \mathbb{T} s < t\}$. Thus, a point $t \in \mathbb{T}$ which is not a maximum of $\mathbb{T}$ is called *right-scattered* if $\sigma(t) > t$, *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$. Finally, by $\mathbb{T}^\kappa$ we denote the set $\{t \in \mathbb{T} : t \text{ is no isolated maximum of } \mathbb{T}\}$ and by $\mu^*(t) := \mu(\sigma(t), t)$ the *graininess* $\mu^* : \mathbb{T} \to [0, \infty)$ of $\mathbb{T}$.

Looking for examples of measure chains it is apparent that subsets of the real line together with the usual ordering $\leq$ and $\mu(s, r) := s - r$ are good candidates. The question which of those sets are in fact measure chains is answered as follows (see Hilger [8, Theorem 1.5.1]):

**Theorem 2.1** *A subset $\mathbb{T}$ of $\mathbb{R}$ is a measure chain (with respect to the usual ordering $\leq$ of $\mathbb{R}$ and $\mu(s, r) = s - r$) if and only if $\mathbb{T}$ has the form $\mathbb{T} = I \setminus O$ where $I$ is an interval and $O$ an open subset of $\mathbb{R}$.*

Thus, real *intervals* and *closed subsets* of $\mathbb{R}$ are important examples of measure chains. On the other hand, they are more than just *examples*, they are to some extent representative. In order to see this, the notion of an isomorphism between measure chains can be introduced by saying that two measure chains $T_1$ and $\mathbb{T}_2$ with respective growth calibrations $\mu_1$ and $\mu_2$ are *isomorphic* if there exists a bijection $f : \mathbb{T}_1 \to \mathbb{T}_2$ such that $\mu_2(f(s), f(r)) = \mu_1(s, r)$ for all $r, s \in \mathbb{T}_1$. With this notion at hand, the following can be said (see Hilger [8, Theorem 2.1]).

**Theorem 2.2** *Each measure chain is isomorphic to a set of the form $I \setminus O$ where $I$ is a real interval and $O$ an open subset of $\mathbb{R}$.*

It is due to this theorem that measure chains are usually considered to be subsets of $\mathbb{R}$ and, even more, that they are *closed* subsets of $\mathbb{R}$. In fact, in most of the literature a measure chain is by definition a closed subset of $\mathbb{R}$ and, as such, it is called a *time scale*. While this is legitimate and (perhaps) helpful for visualizing measure chains, from the point of view of mathematical aesthetics the original definition seems more appropriate.

Before we dwell on the topic of this paper we introduce a means to classify measure chains according to their orders being dense or not (see Neidhart [12, Definition 19]). This distinction is needed in Section 5 on the Cauchy-Riemann integral.

**Definition 2.3** *A measure chain $(\mathbb{T}, \leq, \mu)$ is called* densely ordered *if for any two points $x, z \in \mathbb{T}$ there exists a point $y \in \mathbb{T}$ such that $x \leq y \leq z$ and $x \neq y$ as well as $y \neq z$.*

The following theorem is proved in Neidhart [12, Theorem 80].

**Theorem 2.4** *A measure chain is densely ordered if and only if it is isomorphic to a real interval.*

This theorem tells us that the densely ordered measure chain are, in a way, trivial. The reason is, that the calculus on measure chains reduces to the ordinary calculus of real numbers if the underlying measure chain is isomorphic to an interval.

## 3 The Cauchy Integral

The notion of integral which is commonly used in the literature on measure chains or time scales is the one introduced by Hilger [7], so-called Cauchy-Integral. In order to recall its definition we have to introduce two notions. To this end let $\mathbb{T}$ be an arbitrary measure chain and $\mathcal{Y}$ an arbitrary real or complex Banach space.

**Definition 3.1** *A function $f : \mathbb{T} \to \mathcal{Y}$ is called* regulated *if its left-sided limits exist in all left-dense points of $\mathbb{T}$ and its right-dense limits exist in all right-dense points of $\mathbb{T}$.*

**Definition 3.2** *A function $F : \mathbb{T} \to \mathcal{Y}$ is a called a* pre-antiderivative *of a function $f : \mathbb{T} \to \mathcal{Y}$ if $F$ is continuous and if there exists a set $D \subseteq \mathbb{T}^\kappa$ such that $\mathbb{T}^\kappa \setminus D$ is countable, contains no right-scattered points and has the property that the restriction of $F$ to $D$ is differentiable with derivative $f$.*

The crucial relation between regulated functions and pre-antiderivatives is described in the following theorem (see Hilger [8, Theorem 4.2]).

**Theorem 3.3** *If $f : \mathbb{T} \to \mathcal{Y}$ is a regulated function, then there exists at least one pre-antiderivative $F : \mathbb{T} \to \mathcal{Y}$ of $f$. Moreover, for any two $a, b \in \mathbb{T}$ the difference $F(a) - F(b)$ does not depend on the choice of $F$.*

With this theorem at hand it is straightforward to define an integral for regulated functions.

**Definition 3.4** *For any regulated function $f : [a, b] \to \mathcal{Y}$ from an interval of a measure chain $\mathbb{T}$ to a Banach space $\mathcal{Y}$, the* Cauchy integral *is defined by*

$$\int_a^b f(x)\,\Delta x \;:=\; F(b) - F(a)$$

*where $F : [a, b] \to \mathcal{Y}$ is any pre-antiderivative of $f$ on $[a, b]$.*

Since the Cauchy integral is widely known and extensively used in the literature we do not dwell on it any further. We rather want to mention that the main <u>advantage</u> of this integral is its simplicity, i.e., its simple derivation from the concept of differentiation. No "construction" of the integral by means of some limiting process is necessary. On general measure chains, however, this advantage is somewhat obscured by the fact that pre-antiderivatives are involved, objects which are quite subtle and not easy to handle.

The main <u>disadvantage</u> of the Cauchy integral is, that the set of functions which are integrable, i.e., the set of regulated functions, is too small. In fact, as known from ordinary calculus, the set of regulated functions is even a proper subset of the set of Riemann integrable functions, not to mention the set of Lebesgue integrable functions.

# 4 The Riemann integral

In ordinary calculus, the Riemann integral of a real-valued function is usually defined by either using *Riemann* sums or upper and lower *Darboux* sums. If the values of the function under consideration lie in an arbitrary Banach space, however, the Darboux sum approach cannot be used because of the lacking order structure of a general Banach space. Since, in this paper, we deal with Banach space-valued functions, we therefore mimic the Riemann sum approach.

To this end we first explain what we mean by saying that, for some $\delta > 0$, *a partition* $Z = (a_0, \ldots, a_n)$ of the interval $[a, b]$ *is finer than* $\delta$. In fact, we mean that for each $i = 1, \ldots, n$ we have

$$\text{either} \quad \mu(a_i, a_{i-1}) \leq \delta \quad \text{or both} \quad \mu(a_i, a_{i-1}) > \delta \;\text{ and }\; a_i = \sigma(a_{i-1}).$$

With this notion at hand we can define the integrability of a Banach space-valued function in the sense of Riemann.

**Definition 4.1** *A function $f : [a, b] \to \mathcal{Y}$ from an interval $[a, b]$ of a measure chain $\mathbb{T}$ to Banach space $\mathcal{Y}$ is called* Riemann integrable *if there exists a $y \in \mathcal{Y}$ such that for any $\varepsilon > 0$ there is a $\delta > 0$ with the following property: For any partition $Z = (a_0, \ldots, a_n)$ of $[a, b]$ which is finer than $\delta$ and any set of points $y_1, \ldots, y_n \in \mathcal{Y}$ with $y_j \in [a_{j-1}, a_j)$ for $j = 1, \ldots, n$ one has*

$$\left\| y - \sum_{j=1}^{n} f(y_j)\, \mu(a_j, a_{j-1}) \right\| \;\leq\; \varepsilon\,.$$

Using the fact that for any $\delta > 0$ there always exists a partition of $[a, b]$ which is finer than $\delta$, we easily get the following result.

**Theorem 4.2** *If $f : [a, b] \to \mathcal{Y}$ is Riemann integrable according to Definition 4.1, then the $y \in \mathcal{Y}$ appearing in this definition is uniquely determined, and it is called the* Riemann integral *of $f$, in signs*

$$y \;=\; \int_a^b f(x)\, dx\,.$$

The <u>advantage</u> of the Riemann integral over the Cauchy integral is that the set of Riemann integrable functions is definitively larger than the set of Cauchy integrable functions, i.e. the set of regulated functions. In fact, from ordinary calculus we know (see, e.g., Elstrodt [6, IV, Theorem 6.1]), that a real- or complex-valued bounded function is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous.

The various <u>disadvantages</u> of the Riemann integral are well known from ordinary calculus. Apart from the fact that the set of Riemann integrable functions is too small (compared to the set of Lebesgue integrable functions) we just mention the lack of reasonable convergence results.

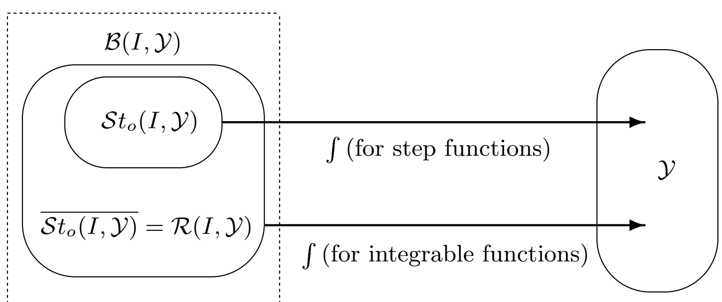
For more on the Riemann integral on time scales we refer to Sailer [14], Guseinov and Kaymakçalan [9], Bohner and Peterson [4, Chapter 5].

# 5 The Cauchy-Riemann integral

In ordinary calculus, the Cauchy-Riemann integral (cf. Amann and Escher [1, Chapter VI.3]) is defined for functions $f : I \to \mathcal{Y}$ where $I$ is a compact interval and $\mathcal{Y}$ a Banach space. The idea underlying this concept of integral is, first to assign to any step function (more precisely, to any function which is constant on any *open* interval of a partition of $I$; we call such a function therefore an *o-step function*) an integral value, and then to extended the set of integrable functions from the Banach space $St_o(I, \mathcal{Y})$ of o-step functions to the Banach space $\mathcal{R}(I, \mathcal{Y})$ of regulated functions. This extension is based on the *Linear Extension Theorem* of Functional Analysis (cf. [1, Theorem 2.6] or [11, Theorem 3.1]) which reads as follows:

**Theorem 5.1** *Let $\mathcal{X} \neq \{0\}$ and $\mathcal{Y}$ be normed linear spaces and $\mathcal{D} \neq \{0\}$ a subspace of $\mathcal{X}$ which is dense in $\mathcal{X}$. Then, for any bounded linear operator $A : \mathcal{D} \to \mathcal{Y}$ there is exactly one bounded linear operator $\overline{A} : \mathcal{X} \to \mathcal{Y}$ which is an extension of $A$. Moreover, for all $x_0 \in \mathcal{X}$ one has $\overline{A}(x_0) = \lim_{x \to x_0} A(x)$.*

This theorem can be applied to the setting where (within the Banach space $\mathcal{B}(I, \mathcal{Y})$ of bounded functions from $I$ to $\mathcal{Y}$) one has $\mathcal{D} = \mathcal{S}t_o(I, \mathcal{Y})$, $\mathcal{X} = \mathcal{R}(I, \mathcal{Y})$ and $A$ is the integral operator which assigns to each o-step function its canonical integral value. This situation is depicted in the following diagram.



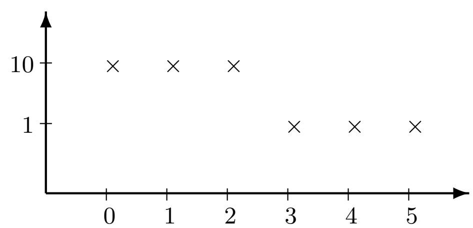

In order to carry over this situation from ordinary to the measure chain calculus we call a function $f$ from an interval $I = [a, b]$ of a measure chain $\mathbb{T}$ to a Banach space $\mathcal{Y}$ an *o-step function* if there exists a partition $Z = (a_0, \ldots, a_n)$ of $[a, b]$ such that the restriction of $f$ to any of the *open* intervals $(a_{i-1}, a_i)$, $i = 1, \ldots, n$, is constant. The set of o-step functions $f : I \to \mathcal{Y}$ is denoted by $St_o(I, \mathcal{Y})$, and the canonical integral operator $\int_{(Z)} : \mathcal{S}t_o(I, \mathcal{Y}) \to \mathcal{Y}$ is defined by means of

$$\int_{(Z)} f := \sum_{i=1}^{n} m_i \, \mu(a_i, a_{i-1}) \tag{1}$$

where $m_i$ is the constant value of $f$ on the open interval $(a_{i-1}, a_i)$ and $\mu(\cdot, \cdot)$ the growth calibration of $\mathbb{T}$.

If $\mathbb{T}$ is densely ordered, it can be shown along the lines of ordinary calculus that the value (1) is independent of the choice of the partition $Z$. However, if $\mathbb{T}$ is not densely ordered, the integral value (1) may change with the partition, and even may fail to be well defined (if at least one of the open intervals $(a_{i-1}, a_i)$ is empty). That, in fact, one may have $\int_{(Z)} f \neq \int_{(U)} f$ for different partitions $Z$ and $U$ can be seen by means of the following simple example.

**Example 5.2** On the measure subchain $\{0, 1, 2, 3, 4, 5\}$ of $\mathbb{R}$ we consider the real-valued function $f$ whose values are $f(0) = f(1) = f(2) = 10$ and $f(3) = f(4) = f(5) = 1$. Then $f$ is an o-step function with respect to the two partitions $Z := (0, 2, 5)$ and $U := (0, 3, 5)$, and the corresponding integral values $\int_{(Z)} f = 2 \cdot 10 + 3 \cdot 1 = 23$ and $\int_{(U)} f = 3 \cdot 10 + 2 \cdot 1 = 32$ are different.




If, in contrast to the previous example, a measure chain is densely ordered, then the Cauchy-Riemann integral can be introduced as in ordinary calculus using the Extension Theorem 5.1. A summary of this approach using o-step functions is as follows (see Neidhart [12, Section 8.1]).

**Theorem 5.3** *If $\mathbb{T}$ is a densely ordered measure chain and $f : [a, b] \to \mathcal{Y}$ an o-step function from an interval $[a, b] \subseteq \mathbb{T}$ to a Banach space $\mathcal{Y}$, then the integral (1) is well-defined and independent of the partition $Z$ of $\mathbb{T}$. Moreover, for any function $f \in \overline{\mathcal{S}t_o(I, \mathcal{Y})}$ the Cauchy-Riemann integral is defined as*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_{(Z)} f_n \tag{2}$$

*where $(f_n)_{n\in\mathbb{N}}$ is any sequence of o-step functions which converges to $f$ uniformly on $[a, b]$.*

*On the other hand, if $\mathbb{T}$ is not densely ordered, the integral (1) may not exist or may depend on the choice of the partition $Z$, and hence the limit (2) may not exist.*

The difficulties with the o-step functions (whose use was motivated from ordinary calculus) can be avoided if one modifies the kind of step functions being used. To this end we introduce the notion of an *h-step function* for any function $f$ from an interval $I = [a, b]$ of a measure chain $\mathbb{T}$ to a Banach space $\mathcal{Y}$, if there exists a partition $Z = (a_0, \ldots, a_n)$ of $[a, b]$ such that $f$ is constant on any *half-open* interval $[a_{i-1}, a_i)$, $i = 1, \ldots, n$, of the partition $Z$.

The set of h-step functions $f : I \to \mathcal{Y}$ is denoted by $St_h(I,\mathcal{Y})$, and the canonical integral operator $\int_{[Z)} : \mathcal{S}t_h(I,\mathcal{Y}) \to \mathcal{Y}$ is defined by

$$\int_{[Z)} f := \sum_{i=1}^{n} m_i\, \mu(a_i, a_{i-1}) \tag{3}$$

where, again, $m_i$ is the constant value of $f$ on $[a_{i-1}, a_i)$ and $\mu(\cdot,\cdot)$ the growth calibration of $\mathbb{T}$.

At first glance, the advantage of h-step functions over o-step functions is not apparent. However, one can show that for any measure chain $\mathbb{T}$ (densely ordered or not) the integral (3) is well defined and independent of the partition $Z$ of $I$. Moreover, any h-step function is also an o-step function, and if the integral $\int_{(Z)} f$ exists, it coincides with $\int_{[Z)} f$. In any case, an application of Theorem 5.1 yields the following result (see Neidhart [12, Section 8.1]).

**Theorem 5.4** *For any measure chain $\mathbb{T}$ and any h-step function $f$ from an interval $I = [a,b] \subseteq \mathbb{T}$ to a Banach space $\mathcal{Y}$ the integral (3) is well-defined and independent of the partition $Z$ of $I$. Moreover, for any function $f \in \overline{\mathcal{S}t_h(I,\mathcal{Y})}$ the Cauchy-Riemann integral is defined as*

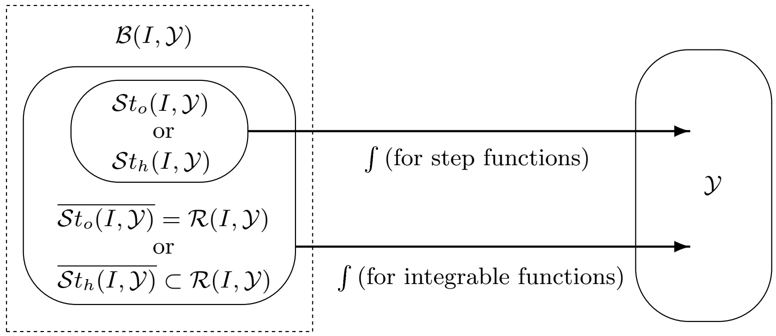
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_{[Z)} f_n \tag{4}$$

*where $(f_n)_{n\in\mathbb{N}}$ is any sequence of h-step functions which converges to $f$ uniformly on $[a,b]$.*

Comparing Theorems 5.3 and 5.4 one might come to the conclusion that the o-step functions approach to the Cauchy-Riemann integral could be dismissed in favor of the h-step functions approach, because the former requires a densely order measure chain while the latter one works for *all* measure chains. This guess, however, is not quite true. The reason is, that in either case the set of integrable functions is determined by an application of the Linear Extension Theorem 5.1 (where $\mathcal{X}$ is the Banach space $\mathcal{B}(I,\mathcal{Y})$ of bounded functions from $I$ to $\mathcal{Y}$, equipped with the sup-norm). In the first case this extension leads from the normed linear space $\mathcal{D} := \mathcal{S}t_o(I,\mathcal{Y})$ to its closure $\overline{\mathcal{S}t_o(I,\mathcal{Y})}$, and in the second case from $\mathcal{D} := \mathcal{S}t_h(I,\mathcal{Y})$ to $\overline{\mathcal{S}t_h(I,\mathcal{Y})}$. The point now is, that $\overline{\mathcal{S}t_o(I,\mathcal{Y})}$ is identical with the Banach space $\mathcal{R}(I,\mathcal{Y})$ of regulated functions, while $\overline{\mathcal{S}t_h(I,\mathcal{Y})}$ can be a proper subspace of $\mathcal{R}(I,\mathcal{Y})$. In any case, however, $\overline{\mathcal{S}t_h(I,\mathcal{Y})}$ contains all $rd$-continuous (and hence all continuous) functions (see Neidhart [12, Theorem 149]).

In summary, we can say that on densely ordered measure chains one should use the o-step function approach. This yields the full set of regulated functions $\mathcal{R}(I,\mathcal{Y})$ as the set of integrable functions while the h-step function approach may yield a proper subset. On measure chains which are not densely ordered, however, the h-step function approach must be used, because the o-step function approach may fail. But then the set of integrable functions may be a proper subset of $\mathcal{R}(I,\mathcal{Y})$.

The overall picture of this situation is given in the following diagram.




The main advantage of the Cauchy-Riemann integral is its quick and elegant introduction. In the context of general measure chains, however, this advantage is obscured by the fact that one has to distinguish between densely ordered measure spaces and those which are not densely ordered, and such a distinction contradicts the general philosophy of the calculus of measure chains. The main disadvantage of the Cauchy integral, however, is, that the set of integrable functions is too small. In fact, it is a (possibly even proper) subset of the set of regulated functions, which in turn is definitively smaller than the set of Riemann integrable functions, not to mention the Lebesgue integrable functions.

# 6 Measure and integral on measure chains

As the term "measure chain" already indicates, and as Hilger mentioned in [7, page 12] (see also [8, page 25]), the growth calibration of a measure chain $\mathbb{T}$ induces a measure on $\mathbb{T}$ in a canonical way. On the other hand, knowing a measure on $\mathbb{T}$, the construction of an integral for functions $f : \mathbb{T} \to [-\infty, +\infty]$ is a straightforward task of measure theory. One may wonder that these facts have not been observed and picked up at the early states of the usage of measure chains, and that still today the Cauchy integral is the standard integral on measure chains.

In the remainder of this paper we sketch, how the construction of the measure theoretic integral indeed works for functions from an arbitrary measure chain $\mathbb{T}$ to an arbitrary real or complex Banach space $\mathcal{Y}$.

To this end we first generate a suitable $\sigma$-Algebra over $\mathbb{T}$ by noticing that each measure chain is a topological space, generated by the open intervals, and that in each topological space the set of open sets generates a $\sigma$-Algebra, the so-called *Borel $\sigma$-Algebra* $\mathfrak{B}$. In order to construct a measure on $\mathfrak{B}$ which is compatible with the growth calibration $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, we introduce a suitable generator $\mathfrak{I}$ of $\mathfrak{B}$ (see Neidhart [12, Theorems 332 and 333]).

**Theorem 6.1** *The system of sets* $\mathfrak{I} := \big\{ [a,b) : a,b \in \mathbb{T}\,,\, a \le b \big\} \cup M$*, where*

$$M \;=\; \begin{cases} \quad\emptyset \quad, & \textit{if } \mathbb{T} \textit{ has no Maximum,} \\ \{\max\mathbb{T}\}, & \textit{otherwise,} \end{cases}$$

*is a semi-ring over* $\mathbb{T}$ *and a generator of* $\mathfrak{B}$.

That the semi-ring $\mathfrak{I}$ is indeed the right choice for our purpose can be seen from the following result (see Neidhart [12, Theorem 338]):

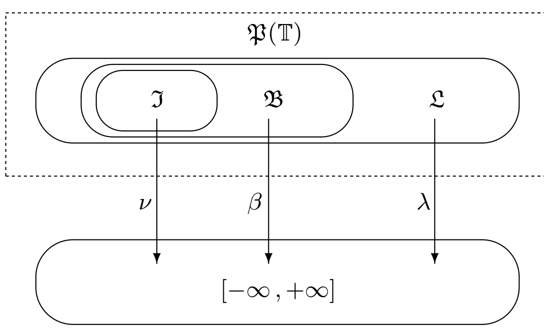
**Theorem 6.2** *The mapping* $\nu : \mathfrak{I} \to \mathbb{R}$ *defined by*

$$\nu(A) \;:=\; \begin{cases} \mu(r,s)\,, & \textit{if } A = [s,r), \textit{ where } s,r \in \mathbb{T} \textit{ and } s \le r, \\ \quad 0 \quad, & \textit{if } \max\mathbb{T} \textit{ exists and } A = \{\max\mathbb{T}\}, \end{cases}$$

*is a* $\sigma$*-finite pre-measure on* $\mathfrak{I}$.

In view of the Extension Theorems of measure theory (see, e.g., Cohn [5], Elstrodt [6]), the pre-measure $\nu$ can be extended to a uniquely determined measure on $\mathfrak{B}$, the so-called *Borel measure* $\beta$. Moreover, if $\mathfrak{L}$ denotes the measure theoretic completion of $\mathfrak{B}$ (i.e. the union of $\mathfrak{B}$ with the set of all subsets of null-sets of $\mathfrak{B}$), then $\mathfrak{L}$ is a $\sigma$-Algebra as well, the so-called *Lebesgue* $\sigma$*-Algebra*, and there exists exactly one extension of $\beta$ (and thus of $\nu$) to a measure $\lambda$ on $\mathfrak{L}$, the so-called *Lebesgue measure*.

The situation we have gained so far is depicted in the following diagram where $\mathfrak{P}(\mathbb{T})$ denotes the set of all subsets of $\mathbb{T}$.




Having the measure spaces $(\mathbb{T}, \mathfrak{B}, \beta)$ and $(\mathbb{T}, \mathfrak{L}, \lambda)$ at hand we are now in a position to introduce the Borel and the Lebesgue integral for functions from a measure chain to a Banach space by simply employing the standard procedure from measure theory. Since this is commonly known only for *real-valued* functions, we first make a brief excursion to the so-called *Bochner integral* which is the proper notion of integral for *Banach space-valued* functions. For more details on the construction of this kind of integral we refer to Cohn [5, Appendix E] (see also Aulbach and Wanner [2, Appendix A]).

Let $(X, \mathfrak{A}, \alpha)$ be a measure space over an arbitrary set $X$ and $\mathfrak{B}(\mathcal{Y})$ the Borel $\sigma$-Algebra of a Banach space $\mathcal{Y}$. Then a function $f : X \to \mathcal{Y}$ is called *measurable* if $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}(\mathcal{Y})$, and a measurable function is called *simple* if it attains only finitely many values. Roughly speaking, the introduction of an integral for measurable functions $f : X \to \mathcal{Y}$ is to first define the integral for simple functions (in the obvious way) and then to define the integral of $f$ by suitably approximating $f$ by a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions and to define the integral of $f$ as the limit of the integrals of the $f_n$.

In case $\mathcal{Y}$ equals $\mathbb{R} \cup \{\pm\infty\}$ or $\mathbb{C}$, this is the standard procedure which can be found in any textbook on measure theory. If $\mathcal{Y}$ is a (real or complex) Banach space with <u>finite</u> dimension, one may choose a basis $\{y_1, \dots, y_d\}$ of $\mathcal{Y}$ and define the (real- or complex-valued, respectively) coordinate mappings $f_i$ of $f$ through the relation $f(x) =: \sum_{i=1}^{d} f_i(x) \cdot y_i$ for all $x \in X$. By means of the relation

$$\int_X f(x)\, d\mu \;:=\; \sum_{i=1}^{d} \Big( \int_X f_i(x)\, d\mu \Big) \cdot y_i$$

the definition of the integral of $f$ is then reduced to the well known integral for real- or complex-valued functions. If the Banach space $\mathcal{Y}$ is <u>infinite</u>-dimensional, however, the general construction of the integral breaks down if the image of $X$ under $f$ is not *separable*, i.e. if $f(X)$ does not contain a dense countable subset. This can be seen as follows:

- If $(f_n)_{n\in\mathbb{N}}$ is a sequence of simple functions $f_n : X \to \mathcal{Y}$ converging to $f : X \to \mathcal{Y}$, then $f(X)$ is contained in the closure of the set $\bigcup_{n=1}^{\infty} f_n(X)$. And since the functions $f_n$ are simple, the sets $f_n(X)$ are finite, thus $\bigcup_{n=1}^{\infty} f_n(M)$ is countable. Consequently, $f(X)$ is necessarily separable.

In addition, in the infinite-dimensional context the following problem arises:

- The sum of two measurable functions from $X$ to $\mathcal{Y}$ is not necessarily measurable (see Cohn [5, Appendix E, Exercise 2]).

In order to overcome these complications one has to replace the class of measurable functions by a more adequate class. In fact, to suitably strengthen the notion of measurability, a function $f : X \to \mathcal{Y}$ is called *strongly measurable* if it is measurable and if $f(X)$ is separable. This definition immediately implies that every simple and every measurable function is strongly measurable. On the other hand, if $\mathcal{Y}$ is a separable Banach space (which is particularly true if $\mathcal{Y}$ is finite-dimensional) then strong measurability is the same as measurability.

Employing the concept of strong measurability, the introduction of an integral for functions from a measure space $(X, \mathfrak{A}, \alpha)$ to a Banach space $(\mathcal{Y}, |\cdot|)$ is almost straightforward. In fact, the set of strongly measurable functions is a linear space (with the usual operations) which is closed under the formation of pointwise limits, and any strongly measurable function is the pointwise limit of a sequence of simple functions. A strongly measurable function $f : X \to \mathcal{Y}$

is then called *(Bochner) integrable*, if the function $|f| : X \to \mathbb{R}$ is integrable with respect to the measure $\alpha$, and the integral of $f$ is defined by means of an approximating sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions as follows: Each $f_n$ can be written in the canonical form $\sum_{j=1}^{k} a_j \, \chi_{A_j}$, and hence its integral is defined as $\int_X f_n(x)\, d\alpha := \sum_{j=1}^{k} a_j \, \alpha(A_j)$. Then, for an arbitrary strongly measurable function $f : X \to \mathcal{Y}$ the integral is defined as

$$\int_X f(x)\, d\alpha := \lim_{n\to\infty} \int_X f_n(x)\, d\alpha\,,$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence of simple functions converging to $f$. That this integral is indeed well defined follows as in the case of real-valued functions. Of the other properties of this integral we just mention that Lebesgue's Dominated Convergence Theorem looks the same as for real-valued functions, while Beppo Levi's Monotone Convergence Theorem is, of course, not available due to the lacking order structure of general Banach spaces. For more properties of the Bochner integral we refer to Cohn [5, Appendix E].

Returning from general measure spaces $(X, \mathfrak{A}, \alpha)$ to the measure spaces $(\mathbb{T}, \mathfrak{B}, \beta)$ and $(\mathbb{T}, \mathfrak{L}, \lambda)$ appearing in the calculus of measure chains, for any function $f$ from an arbitrary measure chain $\mathbb{T}$ to an arbitrary real or complex Banach space $\mathcal{Y}$ we immediately obtain the *Borel* and the *Lebesgue integral*

$$\int_{\mathbb{T}} f(x)\, d\beta \quad \text{and} \quad \int_{\mathbb{T}} f(x)\, d\lambda$$

by simply applying the above-mentioned general result. As a first relation to the previously described integrals we get (see Neidhart [12, Theorem 349]):

**Theorem 6.3** *Any regulated function $f : \mathbb{T} \to \mathcal{Y}$ is Borel and Lebesgue integrable.*

Basically, the definition of the Borel and Lebesgue integral applies to functions which are defined throughout the *whole* measure chain under consideration, and so the (seemingly redundant) question arises of how to define the Borel and the Lebesgue integral for functions which are defined on subsets, in particular intervals, of a measure chain only. In order to answer this question we take two points $a$ and $b$ of a measure chain $\mathbb{T}$ with $a < b$ and notice that the intervals $[a, b]$, $(a, b)$, $[a, b)$ and $(a, b]$ are Borel and Lebesgue measurable. Thus, using the suitable restrictions of $f$ the following integral values are well defined:

$$\int_{[a,b]} f(x)\, d\beta, \quad \int_{(a,b)} f(x)\, d\beta, \quad \int_{[a,b)} f(x)\, d\beta, \quad \int_{(a,b]} f(x)\, d\beta,$$
$$\int_{[a,b]} f(x)\, d\lambda, \quad \int_{(a,b)} f(x)\, d\lambda, \quad \int_{[a,b)} f(x)\, d\lambda, \quad \int_{(a,b]} f(x)\, d\lambda.$$

While in real calculus those terms coincide (if $f$ is Lebesgue integrable), this is not the case in general measure chains. This is due to the following result (see Neidhart [12, Theorem 341]).

**Theorem 6.4** *Any singleton $\{t\} \subseteq \mathbb{T}$ is Lebesgue measurable, and we have*

$$\lambda\big(\{t\}\big) \;=\; \beta\big(\{t\}\big) \;=\; \mu^*(t)\,.$$

*Hence, the Lebesgue and the Borel measure of a singleton $\{t\}$ is 0 for right-dense points $t$, while for right-scattered $t$ it has the positive value of the graininess $\mu^*(t)$.*

Due to this theorem the question arises which of the four Borel and which of the four Lebesgue integrals is suitable for the definition of the respective integral between $a$ and $b$. It turns out (see Neidhart [12, Section 10.2]) that in both cases the use of the half-open interval $[a, b)$ is the choice which leads to the desired result (Theorem 6.5 below). We thus define for any $a, b \in \mathbb{T}$ with $a < b$ and any Borel or Lebesgue integrable Function $f : \mathbb{T} \to \mathcal{Y}$ the *Borel* and the *Lebesgue integral* by

$$\int_a^b f(x)\,d\beta \;:=\; \int_{[a,b)} f(x)\,d\beta \quad \text{and} \quad \int_a^b f(x)\,d\lambda \;:=\; \int_{[a,b)} f(x)\,d\lambda\,,$$

respectively. With these notions at hand we finally get the following result which relates the various notions of integrals considered in this paper (see Neidhart [12, Theorem 350]).

**Theorem 6.5** *Suppose $\mathbb{T}$ is any measure chain, $\mathcal{Y}$ any real or complex Banach space and $f : \mathbb{T} \to \mathcal{Y}$ strongly measurable. Then, if for some $a, b \in \mathbb{T}$ with $a < b$ the restriction of $f$ to the interval $[a, b)$ is Lebesgue integrable, it is also integrable in the sense of Cauchy, Riemann, Cauchy-Riemann and Borel, and the corresponding integrals have the same value as $\int_a^b f(x)\,d\lambda$.*

We close this section by noticing that the Lebesgue integral for real-valued functions on time scales has recently been considered by Bohner and Guseinov in Bohner and Peterson [4, Chapter 5].

# 7 Conclusion: The best is (almost) for free

We summarize our previous considerations by stating that — as in ordinary calculus — the Lebesgue integral is by far superior to all other notions of integrals which are possible on measure chains. The Lebesgue integral does not only provide the largest set of integrable functions, its derivation is even simpler than the construction of the other types of integrals, because most of the technical details can be avoided by simply quoting standard results from measure theory. In fact, once it is observed that the growth calibration of a measure chain canonically generates a $\sigma$-finite pre-measure on the semi-ring of half-open intervals, the rest of the work is done by standard Extension Theorems. We therefore come to the astonishing conclusion that in the context of integration on measure chains the best is (almost) for free.

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